

Nonelectromagnetic duality in twisted $N = 4$ model

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Abstract

In this paper we discuss the possible existing correlation functions in the $N = 4$ topological model. Due to the distinguished feature that no anomaly exists in $N = 4$ supersymmetric theories, the positive-negative ghost number balance has to be taken into account while considering the correlation functions. On restriction to Kähler manifolds we may find a perturbative mass term which breaks the $N = 4$ supersymmetry down to $N = 1$. In all of these, a non-electromagnetic duality plays an important role. Moreover, to get a computable generating functional the existence of a proper vanishing theorem is required.

Key Words: $N = 4$ twisted model, topological field, correlation function.

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1 Introduction

Topological quantum field theories [1] have been pushed forward vigorously during these years because of the celebrated Seiberg-Witten theory [2, 3]. This beautiful theory achieves the strong-weak duality in the $N = 2$ supersymmetric model on the one hand and provides a powerful tool for testing the differential topological structure of a manifold on the other [4].

Among topological field theories the correlation function which represents the Donaldson invariant is one of the essentials [5]. However, at least for the gauge group $SU(2)$, known topological observables are only those represented by correlation functions of the fields $\text{Tr}(\phi^2)$ (in which ϕ is a field of ghost number +2) and its descendent k -forms ($0 < k \leq 4$), even when matter fields are presented [6]. For $N = 4$ model [7], in which the anomaly-free feature is a well known fact [8], one may naively think that no nonvanishing correlation functions (in vacuum) exist because any matrix element of an operator of positive ghost number vanishes. However, the $N = 4$ model is a larger model including more fields in its multiplets. Can we expect some suitable operators of negative ghost number to balance the positive ghost numbers and hence constitute some non-vanishing correlation functions as topological observables? The answer is affirmative. In fact we have a nonelectromagnetic duality in the underlying model under which the fields of positive and negative ghost numbers are dual to each other. This makes it possible to construct nonvanishing correlation functions between pairs of dual fields with opposite ghost numbers.

By using the Mathai-Quillen formalism [9] we have given the action of $N = 4$ Yang-Mills model and related BRST and anti-BRST transformations [10]. In this paper we would like to discuss the duality symmetry in more detail and use this symmetry to analyze the topological polynomial invariants in $N = 4$ model.

This paper is organized as follows. In section 2 we make a brief review about the twisted supersymmetric $N = 4$ Yang-Mills theory with emphasis on the description of nonelectromagnetic duality symmetry. Next, we show in section 3 that topological invariants can be constructed from correlation functions of a pair of dual fields with opposite ghost numbers. Section 4 is devoted to the understanding of the correlation functions on Kähler manifold and the reduction to $N = 1$ theory. We will perform a mass term perturbation in section 5 and give the total result in section 6. In the end of the paper we shall supply a short discussion.

2 The action and nonelectromagnetic duality

In ref.[10] the action of $N = 4$ topological model has been derived using the Mathai-Quillen technique with suitable “nonminimal gauge fermion” term. The model we would like to discuss has the twisted $SU(2)_L \times SU(2)'_R \times SU(2)_F$ symmetry [7], where $SU(2)'_R$ is the diagonal contribution of $SU(2)_R \times SU(2)_I$ and $SU(2)_I \times SU(2)_F = SO(4) \subset SU(4)$, the global symmetry group of the $N = 4$ supersymmetric Yang-Mills model (the subscripts I, F are used only for distinguishing the two $SU(2)$ subgroups). So we have the following spectrum of particles:

- Bosons

$$(1/2, 1/2, 0) A_i(0), (0, 1, 0) B_{ij}(0), (0, 0, 1) \phi(2), \tilde{\phi}(-2), C(0);$$

- Fermions

$$(1/2, 1/2, 1/2) \psi_i(1), \tilde{\chi}_i(-1), (0, 1, 1/2) \chi_{ij}(-1), \tilde{\psi}_{ij}(1), \\ (0, 0, 1/2) \eta(-1), \xi(1),$$

where the ghost numbers are marked in the brackets following the field operators, and for clarity we supply that B_{ij} , χ_{ij} and $\tilde{\psi}_{ij}$ are chosen as anti-selfdual fields. With two auxiliary fields

(anti-selfdual) H_{ij} (0), \tilde{H}_i (0) supplemented we can write down the BRST and anti-BRST transformations [10],

$$\begin{aligned}
\delta A^i &= \epsilon^A \psi_A^i, & \psi_1^i &= \psi^i, \psi_2^i = \tilde{\chi}^i, \\
\delta B^{ij} &= \epsilon^A \chi_A^{ij}, & \chi_1^{ij} &= \tilde{\psi}^{ij}, \chi_2^{ij} = \chi^{ij}, \\
\delta \Phi_{AB} &= \frac{1}{2}(\epsilon_A \eta_B + \epsilon_B \eta_A), & \Phi_{11} &= \Phi^{22} = \phi, \Phi_{22} = \Phi^{11} = \tilde{\phi}, \\
& & \Phi_{12} &= \Phi_{21} = -\Phi^{12} = -\Phi^{21} = C, \\
& & \eta_1 &= \xi, \eta_2 = \eta, \\
\delta \psi_A^i &= \epsilon^B (\epsilon_{AB} \tilde{H}^i + D^i \Phi_{AB}), & & \\
\delta \chi_A^{ij} &= \epsilon^B (\epsilon_{AB} H^{ij} + [B^{ij}, \Phi_{AB}]), & & \\
\delta \eta_A &= \epsilon_B [\Phi_{AC}, \Phi^{CB}], & & \\
\delta \tilde{H}^i &= -\epsilon^C (\epsilon^{AB} [\psi_A^i, \Phi_{BC}] + \frac{1}{2} D^i \eta_C), & & \\
\delta H^{ij} &= -\epsilon^C (\epsilon^{AB} [\chi_A^{ij}, \Phi_{BC}] + \frac{1}{2} [B^{ij}, \eta_C]). & &
\end{aligned} \tag{2.1}$$

The topological charges are defined as (f represents any one of the above fields)

$$\delta f = \epsilon^A [Q_A, f], \tag{2.2}$$

and the action we found can be represented as

$$S = \{Q_1, [Q_2, W]\} = -\{Q^2, [Q_2, W]\}, \tag{2.3}$$

where

$$\begin{aligned}
W &= \frac{1}{2e^2} Tr \{ B^{ij} (2iF_{ij}^\dagger - H_{ij}) + \frac{2i\alpha}{3} B_{ij} [B_k^i, B^{jk}] \\
&\quad + \psi^i \tilde{\chi}_i - 2\beta C [\tilde{\phi}, \phi] \}.
\end{aligned} \tag{2.4}$$

Later we will take $\alpha = 1$ by means of untwisted $N = 4$ supersymmetric transformation. Now we find immediately that (2.1) and (2.4) are invariant under the following duality transform,

$$\begin{aligned}
\epsilon^1 &\rightleftharpoons \epsilon^2, \\
i &\rightleftharpoons -i, \\
A^i &\rightleftharpoons A^i, \\
B^{ij} &\rightleftharpoons B^{ij}, \\
C &\rightleftharpoons C, \\
\phi &\rightleftharpoons \tilde{\phi}, \\
\psi^i &\rightleftharpoons \tilde{\chi}^i, \\
\chi^{ij} &\rightleftharpoons \tilde{\psi}^{ij}, \\
\eta &\rightleftharpoons -\xi, \\
\tilde{H}^i &\rightleftharpoons -\tilde{H}^i, \\
H^{ij} &\rightleftharpoons -H^{ij}.
\end{aligned} \tag{2.5}$$

For convenience we will call it Gh-duality, because the dual fields have opposite ghost numbers under this duality.

Using (2.1), eq.(2.3) gives the explicit action density

$$\begin{aligned}
S = \frac{1}{2e^2} \text{Tr} \{ & -(H_{ij} - iF_{ij}^\dagger - i\alpha[B_{ki}, B_j^k])^2 - (\tilde{H}_i - 2iD^j B_{ij})^2 \\
& - (S_{ij})^2 - (k_i)^2 - \chi^{ij}[\phi, \chi_{ij}] - \tilde{\chi}^i[\phi, \tilde{\chi}_i] \\
& - \chi^{ij}(4iD_i\psi_j + 4i\alpha[B_{ki}, \tilde{\psi}_j^k] - 2[C, \tilde{\psi}_{ij}] + [B_{ij}, \xi]) \\
& - \tilde{\chi}^i(4iD^j\tilde{\psi}_{ij} - 4i[B_{ij}, \psi^j] - 2[C, \psi_i] + D_i\xi) \\
& D^i\phi D_i\tilde{\phi} - \psi^i[\tilde{\phi}, \psi_i] + \psi^i D_i\eta \\
& - [B^{ij}, \phi][B_{ij}, \tilde{\phi}] - \tilde{\psi}^{ij}[\tilde{\phi}, \tilde{\psi}_{ij}] + \tilde{\psi}^{ij}[B_{ij}, \eta] \\
& + \beta([\tilde{\phi}, \phi]^2 + 4[C, \tilde{\phi}][C, \phi] + \eta[\phi, \eta] + \xi[\tilde{\phi}, \xi] + 2C\{\eta, \xi\}) \}, \tag{2.6}
\end{aligned}$$

in which ¹

$$\begin{aligned}
S_{ij} &= F_{ij}^\dagger - i\varpi[B_{ij}, C] + \alpha[B_{ki}, B_j^k], \\
k_i &= 2D^j B_{ij} - i\varpi D_i C \tag{2.7}
\end{aligned}$$

are modified sections of vector bundle $\mathcal{M} \times_{\mathcal{G}} \mathcal{V}$ (\mathcal{M} -moduli space of instanton, \mathcal{V} -fiber, \mathcal{G} -gauge transformation group [7, 10]), where ϖ changes from +1 to -1 under the symmetry (2.5). This sign is only superficial because Vafa and Witten have shown that the crossing terms in $(S_{ij})^2$ and $(k_i)^2$ cancels each other. Therefore we can write

$$S = -\{Q^1, [Q_1, W]\} = -\{Q^2, [Q_2, W]\} = -\frac{1}{2}\{Q^A, [Q_A, Q]\}. \tag{2.8}$$

The zero section equations

$$S_{ij} = 0, \quad k_i = 0 \tag{2.9}$$

denote the self dual equation of instantons, if suitable vanishing theorem holds [7].

3 Observables and correlation functions

We first calculate the stress energy tensor. By definition, the stress-energy tensor T_{ij} obey the formula

$$\delta_M L = \delta_M \int_M \sqrt{g} S = \int_M \sqrt{g} \delta_M g^{ij} T_{ij}. \tag{3.1}$$

Since the variation δ_M of the metric is independent of supersymmetry transformation, it commutes with the charges Q_A . Besides note that all of the anti-selfdual fields (χ_{ij} etc.) should subject to a constraint which requires that [1]

$$\delta_M \chi_{ij} = -\frac{i}{2} \epsilon_{ijk'l'} \delta_M g^{k'l} g^{l'l} \chi_{kl} - \frac{1}{4} (\delta_M g^{rs} g_{rs}) \chi_{ij}. \tag{3.2}$$

In particular, this leads to

$$\delta_M (\sqrt{g} \chi_{ij} \psi^{ij}) = 0$$

for any tensor field ψ^{ij} . It is then straightforward to write down the stress-energy tensor

¹Here F_{ij}^\dagger is the anti-selfdual part of $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$, and Vafa and Witten have proved that $(D^j B_{ij})^2$ can further be expressed as a scalar curvature and anti-selfdual part of the Weyl tensor in curved space.

$$T_{ij} = -\frac{1}{2}\{Q^A, [Q_A, V_{ij}]\}, \quad (3.3)$$

in which

$$\begin{aligned} V_{ij} &= \frac{2}{\sqrt{g}} \frac{\delta_M(\sqrt{g}W)}{\delta_M g^{ij}} \\ &= \frac{1}{e^2} \text{Tr}\{\psi_i \tilde{\chi}_j - g_{ij}(\psi^k \tilde{\chi}_k - 2\beta C[\tilde{\phi}, \phi])\}. \end{aligned} \quad (3.4)$$

Witten pointed out that the observables of a topological theory are those operators in the cohomology of related topological charge [1]. In the present case, Witten's results on the Feynman integrations will be generalized to $\langle Q_A, \mathcal{O} \rangle = 0$ for any operator \mathcal{O} . Moreover, due to the peculiar form of our T_{ij} , the variation of a nonvanishing path integral under a change in the metric will be zero if either one of $\{Q_1, \mathcal{O}\} = 0$ or $\{Q_2, \mathcal{O}\} = 0$ holds (we are only interested in those operators which do not depend on g_{ij} explicitly). So, for the operators which are neither explicitly depend on the metric nor Q_A exact we may construct topological invariants. Looking over the transformations (2.1), we can find the candidate fields with ghost numbers +4 and -4 respectively,

$$\begin{aligned} \mathcal{O}_1(x) &\equiv \mathcal{O}_1^{(0)}(x) = \frac{1}{8\pi^2} \text{Tr} \phi^2(x), \\ \mathcal{O}_2(x) &\equiv \mathcal{O}_2^{(0)}(x) = \frac{1}{8\pi^2} \text{Tr} \tilde{\phi}^2(x). \end{aligned} \quad (3.5)$$

Other observables (if any) should be constructed from the descendent operators defined as

$$d\mathcal{O}_A^{(k)} = \{Q_A, \mathcal{O}_A^{(k+1)}\}, \quad A = 1, 2. \quad (3.6)$$

To be explicit, we have

$$\begin{aligned} \mathcal{O}_1^{(1)}(x) &= \frac{1}{4\pi^2} \text{Tr}(\phi\psi)(x), & \mathcal{O}_2^{(1)}(x) &= \frac{1}{4\pi^2} \text{Tr}(\tilde{\phi}\tilde{\chi})(x), \\ \mathcal{O}_1^{(2)}(x) &= \frac{1}{8\pi^2} \text{Tr}(\psi \wedge \psi + 2\phi F)(x), & \mathcal{O}_2^{(2)}(x) &= \frac{1}{8\pi^2} \text{Tr}(\tilde{\chi} \wedge \tilde{\chi} + 2\tilde{\phi} F)(x), \\ \mathcal{O}_1^{(3)}(x) &= \frac{1}{4\pi^2} \text{Tr}(\psi \wedge F)(x), & \mathcal{O}_2^{(3)}(x) &= \frac{1}{4\pi^2} \text{Tr}(\tilde{\chi} \wedge F)(x), \\ \mathcal{O}_1^{(4)}(x) &= \frac{1}{8\pi^2} \text{Tr}(F \wedge F)(x) = \mathcal{O}_2^{(4)}(x) \end{aligned} \quad (3.7)$$

in which $\psi = \psi_i dx^i$, $\tilde{\chi} = \tilde{\chi}_i dx^i$ are one forms, and $F = dA + A^2 = \frac{1}{2}F_{ij}dx^i \wedge dx^j$ is a two form. As a matter of fact, the above operators are the group of observables with positive ghost numbers found in ref.[1] and another group of observables with negative ghost numbers, and both groups of observables are governed by the Gh-duality.

It is also pointed out by Witten that the corresponding BRST invariants are

$$I_A(\Sigma) = \int_{\Sigma} \mathcal{O}_A^{(k)}(x), \quad (3.8)$$

in which Σ is a k -dimensional homology cycle and $I_A(\Sigma)$ depends only on the homology class of Σ . When we study the correlation functions on simple connected four-manifolds M , we have only to consider $k = 0, 2$ invariants. So we may have the general correlation functions

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_1(x_r) I_1(\Sigma_1) \dots I_1(\Sigma_s) \mathcal{O}_2(x_{r+1}) \dots \mathcal{O}_2(x_{r+r'}) I_2(\Sigma_{s+1}) \dots I_2(\Sigma_{s+s'}) \rangle. \quad (3.9)$$

Since there is no anomaly in $N = 4$ supersymmetry, the ghost numbers of observables entering the correlation functions have to be balanced, i.e. we have to impose

$$4r + 2s = 4r' + 2s'. \quad (3.10)$$

The problem is now how to construct the generating functional for the above correlation functions.

Let us study a simple case as an illustrating example. Under the condition discussed in [5] via cluster decomposition, one gets

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_1(x_r) \mathcal{O}_2(x_{r+1}) \dots \mathcal{O}_2(x_{r+r'}) \rangle = \langle \mathcal{O} \rangle_\Omega^r \langle \mathcal{O} \rangle_{\Omega'}^{r'} \langle 1 \rangle, \quad (3.11)$$

in which Ω and Ω' represent different vacua with ghost numbers $+4$ and -4 respectively. The constraint (3.10) leads to $r = r'$. Similar consideration is applicable to $I_A(\Sigma)$. As a whole, we may have

$$\begin{aligned} & \left\langle \exp \sum_{A=1,2} \left(\sum_{a=1}^{s+s'} \alpha_{Aa} I_A(\Sigma_a) + \lambda_A \mathcal{O}_A \right) \right\rangle \\ &= \sum_{\rho} C_{\rho} \exp \left(\sum_{A,B} \frac{\eta_{\rho}^{AB}}{2} \sum_{a,b} \alpha_{Aa} \alpha_{Bb} \#(\Sigma_a \cap \Sigma_b) + \sum_A \lambda_A \langle \mathcal{O}_A \rangle_{\rho} \right) \Big|_0, \end{aligned} \quad (3.12)$$

in which $\#(\Sigma_a \cap \Sigma_b)$ is the intersection number of Σ_a and Σ_b , the vacuum expectation values η and $\langle \mathcal{O} \rangle$, α and λ are universal constants, and the symbol $|_0$ means the restriction (3.10).

Since the generating functional is quite complicated, we would like to calculate its main part of contributions instead. Following the explanations in the end of Section 2, it is clear that the moduli space \mathcal{M} of instantons is a subspace of the moduli space of eq.(2.9). Thus the zero modes corresponding to the moduli space \mathcal{M} cannot include all the fermions in $N = 4$ model. From the transformation law $\delta A^i = \epsilon^A \psi_A^i$, it is easy to realize that there are two tangents to \mathcal{M} we can choose, each corresponds to a group of zero modes.

One may think that the measure for the path integral has equal numbers of zero modes for fermions with positive and negative ghost numbers. We thus can split the integration measure into two parts, one is a measure with positive ghost number and the other is one with negative ghost number. Because of eq.(3.10), the total integration measure should have no zero modes. Therefore, the correlation function in (3.9) could be splitted into the following form under a reasonable good approximation,

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_1(x_r) I_1(\Sigma_1) \dots I_1(\Sigma_s) \rangle_1 \langle \mathcal{O}_2(x_{r+1}) \dots \mathcal{O}_2(x_{r+r'}) I_2(\Sigma_{s+1}) \dots I_2(\Sigma_{s+s'}) \rangle_2, \quad (3.13)$$

in which the action S reduces to a pair of (twisted) $N = 2$ actions with fermions ψ^i , χ^{ij} and η in one of them and $\tilde{\chi}^i$, $\tilde{\psi}^{ij}$ and ξ in the other. Obviously, in doing so we have assumed the existence of vanishing theorem and also a vanishing nonminimal term.

Now assume that there is a mass gap (later on we shall see that a mass gap do generate when we perform a mass term perturbation which breaks the $N = 4$ supersymmetry down to $N = 1$). Then either parts of the correlation function (3.13) can be expressed through a generating functional,

$$\left\langle \exp \left(\sum_a \alpha_{Aa} I_A(\Sigma_a) + \lambda_A \mathcal{O}_A \right) \right\rangle_A = \sum_{\rho} e^{a_{\rho} \chi + b_{\rho} \sigma} \exp \left(\frac{1}{2} \eta_{A\rho} \sum_{a,b} \alpha_{Aa} \alpha_{Ab} \#(\Sigma_a \cap \Sigma_b) + \lambda_A \langle \mathcal{O}_A \rangle_{\rho} \right), \quad (3.14)$$

where χ and σ are Euler characteristic and signature respectively. Notice that the two-point function $\langle I_1(\Sigma_a) I_2(\Sigma_b) \rangle$ would not appear in our approximation.

4 Kähler manifolds and reduction to $N = 1$

To find the formulation on Kähler manifold we have first to write down the untwisted $N = 4$ supersymmetric transformations. But there is no known off-shell formulation without constrained fields [11]. So we start from the on-shell form [12]

$$\begin{aligned}
\delta A^i &= -\frac{1}{2}(\bar{\eta}_{j\dot{\alpha}}\bar{\sigma}^{i\dot{\alpha}\alpha}\psi_\alpha^j + \bar{\psi}_\alpha^j\bar{\sigma}^{i\dot{\alpha}\alpha}\eta_{j\dot{\alpha}}), \\
\delta\Phi^{ij} &= \bar{\eta}_\alpha^i\psi^{\dot{\alpha}j} - \bar{\eta}_\alpha^j\psi^{\dot{\alpha}i} + \epsilon^{ijkl}\eta_k^\alpha\psi_{\alpha l}, \\
\delta\psi_\alpha^i &= \bar{\eta}_j^\alpha D_{\alpha\dot{\alpha}}\Phi^{ij} + \frac{1}{4}\eta^{i\beta}\sigma_{\alpha\beta}^{jk}F_{jk} - \frac{1}{2}\eta_\alpha^j[\Phi^{ik}, \Phi_{kj}^\dagger], \\
\delta\bar{\psi}_\alpha^i &= \frac{1}{4}\bar{\eta}^{i\beta}\bar{\sigma}_{\dot{\alpha}\beta}^{jk}F_{jk} - \frac{1}{2}\bar{\eta}_\alpha^j[\Phi^{ik}, \Phi_{kj}^\dagger] + \eta_j^\alpha D_{\alpha\dot{\alpha}}\Phi^{ij}.
\end{aligned} \tag{4.1}$$

The spinor algebra and related notations are adopted from Wess-Bagger's book [13]. Now, the twisted or topological transformation laws can be obtained by setting

$$\bar{\eta}_{j\dot{\alpha}} = \epsilon^A \sigma_{jA\dot{\alpha}}, \quad \eta_{i\alpha} = 0. \tag{4.2}$$

In fact, using the relations

$$\begin{aligned}
\Phi^{ij} &= iB^{ij} + \frac{1}{2}\sigma_{AB}^{ij}\Phi^{AB} = \frac{i}{2}\bar{\sigma}_{\dot{\alpha}\beta}^{ij}B^{\dot{\alpha}\beta} + \frac{1}{2}\sigma_{AB}^{ij}\Phi^{AB}, \\
\sigma_{i\alpha\dot{\alpha}}A^i &= A_{\alpha\dot{\alpha}}, \\
\sigma_{iA\dot{\alpha}}\psi_\alpha^i &= \psi_{\alpha\dot{\alpha}} = \sigma_{i\alpha\dot{\alpha}}\psi_A^i, \\
\bar{\sigma}_{i\dot{\beta}}^A\bar{\psi}_\alpha^i &= \bar{\psi}_{\alpha\dot{\beta}}^A = \frac{1}{2i}\chi_{\dot{\alpha}\dot{\beta}}^A + \frac{1}{4}\epsilon_{\dot{\alpha}\dot{\beta}}\eta^A = \frac{1}{2i}\bar{\sigma}_{\dot{\alpha}\dot{\beta}}^{ij}\chi_{ij}^A + \frac{1}{4}\epsilon_{\dot{\alpha}\dot{\beta}}\eta^A,
\end{aligned} \tag{4.3}$$

these transformation laws coincide with (2.1) if the auxiliary fields are replaced by the following expressions (see eq.(2.6)),

$$\begin{aligned}
\tilde{H}^i &= 2iD_j B^{ij}, \\
H^{ij} &= i(F^{ij} + \alpha[B_k^i, B^{kj}])
\end{aligned} \tag{4.4}$$

with $\alpha = 1$.

When the metric on M under consideration is Kähler, the holonomy is $SU(2)_L \times U(1)_R$ instead of $SU(2)_L \times SU(2)_R$, the two-dimensional representation of $SU(2)_R$ decomposes under $U(1)_R$ into a sum of two one-dimensional representations. We follow Witten to use type $(0, 1)$ for one forms $dx^m\sigma_{m\alpha\dot{2}}$, type $(1, 0)$ for one forms $dx^m\sigma_{m\alpha\dot{1}}$. Similarly we have (notice that the suffices $\dot{1}$ and $\dot{2}$ are interchanged in our notations as compared to that of Witten)

$$\eta_{i\alpha} = 0, \quad \bar{\eta}_{j\dot{2}} = \rho_1^A \sigma_{jA\dot{2}} \quad \text{and} \quad \bar{\eta}_{j\dot{1}} = \rho_2^A \sigma_{jA\dot{1}}. \tag{4.5}$$

For some reasons argued in ref.[5], we consider here only ρ_1^A (and omit the suffix 1 later) symmetry with $\bar{\eta}_{j\dot{1}} = 0$. The transformation laws are

$$\begin{aligned}
\delta A^{\alpha\dot{1}} &= \rho^A \psi_A^{\alpha\dot{1}}, \\
\delta A^{\alpha\dot{2}} &= 0, \\
\delta\Phi_{AB} &= \frac{1}{4}(\rho_A \eta_B + \rho_B \eta_A),
\end{aligned}$$

$$\begin{aligned}
\delta B^{\dot{1}\dot{1}} &= 0, \\
\delta B^{\dot{1}\dot{2}} &= \frac{1}{2}\rho^A \chi_A^{\dot{1}\dot{2}}, \\
\delta B^{\dot{2}\dot{2}} &= \rho^A \chi_A^{\dot{2}\dot{2}}, \\
\delta \psi_{\alpha A}^{\dot{1}} &= i\rho^A D_{\alpha\dot{1}} B^{\dot{1}\dot{1}}, \\
\delta \psi_{\alpha A}^{\dot{2}} &= -\rho^B D_{\alpha\dot{1}} \Phi_{AB} - i\rho_A D_{\alpha\dot{1}} B^{\dot{1}\dot{2}}, \\
\delta \chi_A^{\dot{1}\dot{1}} &= 2i\delta \bar{\psi}_A^{\dot{1}\dot{1}} = \frac{i}{2}\rho_A [B_{2\gamma}, B^{\gamma\dot{1}}] + \rho^B [B^{\dot{1}\dot{1}}, \Phi_{AB}], \\
\delta \chi_A^{\dot{2}\dot{2}} &= 2i\delta \bar{\psi}_A^{\dot{2}\dot{2}} = i\rho_A \bar{\sigma}_{jk}^{\dot{2}\dot{2}} F^{jk}, \\
\delta(\chi_A^{\dot{1}\dot{2}} + \frac{i}{2}\eta_A) &= 2i\delta \bar{\psi}_A^{\dot{1}\dot{2}} = i\rho_A (\bar{\sigma}_{jk}^{\dot{1}\dot{2}} F^{jk} + \frac{1}{2}[B^{\dot{1}\dot{1}}, B^{\dot{2}\dot{2}}]) \\
&\quad + i\rho^B (\frac{1}{2}[\Phi_{AC}, \Phi_B^C] - i[B^{\dot{1}\dot{2}}, \Phi_{AB}]) \\
\delta(\chi_A^{\dot{1}\dot{2}} - \frac{i}{2}\eta_A) &= 2i\delta \bar{\psi}_A^{\dot{2}\dot{1}} = 0.
\end{aligned} \tag{4.6}$$

Inspecting eq.(4.6) we find that these formulas are almost a double copy of the corresponding $N = 2$ transformation laws (cf.[5] eq.(3.13)) if we put $B^{\dot{\alpha}\dot{\beta}} = C(= \Phi_{12}) = 0$,

$$\begin{aligned}
\delta A^{\alpha\dot{1}} &= \rho^A \psi_A^{\alpha\dot{1}}, \quad \delta A^{\alpha\dot{2}} = 0, \\
\delta \Phi_{AA} &= \frac{1}{2}\rho_A \eta_A, \\
\delta \psi_{\alpha\dot{2}A} &= 0, \\
\delta \psi_{\alpha\dot{1}A} &= \rho^A D_{\alpha\dot{1}} \Phi_{AA}, \\
\delta \bar{\psi}_{\dot{\alpha}\dot{1}}^A &= \frac{1}{2}\rho^A \bar{\sigma}_{\dot{\alpha}\dot{1}}^{jk} F_{jk} - \frac{1}{4}\epsilon_{\dot{\alpha}\dot{1}} \rho^A [\Phi_{AA}, \Phi^{AA}], \\
\delta \bar{\psi}_{\dot{\alpha}\dot{2}}^A &= 0,
\end{aligned} \tag{4.7}$$

where the summation over A appears only in the first equality. The conditions we used are nothing but the vanishing theorem and vanishing nonminimal term, under which the partition function has been represented as the Euler characteristics of instanton moduli spaces [7, 10], and the correlation functions can be divided approximately into two parts as is mentioned in the last section. Vafa and Witten made an exhaustive study of the subject, following whom the existence of vanishing theorem is much convincible for gauge group $SU(2)$ or a product of $SU(2)$'s [7]. Provided vanishing theorem holds we can imitate ref.[5] step by step to get the correlation functions on Kähler manifolds. For example, the (anti)BRST invariance leads to $F^{0,2} = 0$, the holomorphic structure of the bundle is (anti-) BRST invariant, and $Q_1^A \equiv \hat{Q}^A$ corresponding to ρ_1^A is enough in analyzing the topological correlation functions and so on.

However, before studying the mass perturbation we would like to describe the more general pattern on how to reduce an $N = 4$ multiplet to the $N = 1$ multiplets. A tentative scheme is the following,

$N = 4(\text{twisted})$	\longrightarrow	$N = 2$	\longrightarrow	$N = 1$
		Gauge multiplet	$U(1)$	Gauge
$H^{ij} (0, 1, 0)$	(0)	$H^{ij} (0, 1, 0)$	0	$H^{i\dot{2}} (0, 1^0, 0)$
$\tilde{H}^i (1/2, 1/2, 0)$	(0)	$A^i (1/2, 1/2, 0)$	0	$A_{\alpha\dot{\alpha}} (1/2, 1/2, 0)$
$A^i (1/2, 1/2, 0)$	(0)	$\phi, \tilde{\phi} (0, 0, 1^+ \oplus 1^-)$	$2 \oplus -2$	$\psi_{\alpha\dot{2}}^1 \equiv \lambda_{\alpha}^1 (1/2, 1/2^-, 1/2^+)$
$B^{ij} (0, 1, 0)$	(0)	$\chi^{ij} (0, 1, 1/2^-)$	-1	$\left. \begin{matrix} \psi_{\dot{2}2}^2 \\ \psi_{i2}^2 \end{matrix} \right\} \equiv \bar{\lambda}_{\dot{\alpha}}^1 (0, 1^-, 1/2^-)$
$\phi, \tilde{\phi}, C (0, 0, 1)$	(2, -2, 0)	$\psi^i (1/2, 1/2, 1/2^+)$	1	Chiral U
$\chi^{ij}, \tilde{\psi}^{ij} (0, 1, 1/2)$	(-1, 1)	$\eta (0, 0, 1/2^-)$	-1	$H'(H^{i1}, H^{\dot{2}2}) (0, 1^+ \oplus 1^-, 0)$
$\psi^i, \tilde{\chi}^i (1/2, 1/2, 1/2)$	(1, -1)	Hypermultiplet		$\Phi'(\phi, \tilde{\phi}) (0, 0, 1^+ \oplus 1^-)$
$\eta, \xi (0, 0, 1/2)$	(-1, 1)	$\tilde{H}^i (1/2, 1/2, 0)$	0	$\psi_{\alpha\dot{1}}^1 \equiv \psi_{\alpha} (1/2, 1/2^+, 1/2^+)$
		$B^{ij} (0, 1, 0)$	0	$\left. \begin{matrix} \psi_{i1}^2 \\ \psi_{\dot{2}1}^2 \end{matrix} \right\} \equiv \bar{\psi}_{\dot{\alpha}} (0, 1^+, 1/2^-)$
		$C (0, 0, 1^0)$	0	Chiral V
		$\tilde{\psi}^{ij} (0, 1, 1/2^+)$	1	$H''(\tilde{H}_{\alpha\dot{1}}) (1/2, 1/2^+, 0)$
		$\tilde{\chi}^i (1/2, 1/2, 1/2^-)$	-1	$\Phi''(B^{i1}, B^{\dot{2}2}) (0, 1^+ \oplus 1^-, 0)$
		$\xi (0, 0, 1/2^+)$	1	$\psi_{\alpha\dot{1}}^2 \equiv \tilde{\chi}_{\alpha} (1/2, 1/2^+, 1/2^-)$
				$\left. \begin{matrix} \tilde{\psi}_{i1}^1 \\ \tilde{\psi}_{\dot{2}1}^1 \end{matrix} \right\} \equiv \bar{\tilde{\chi}}_{\dot{\alpha}} (0, 1^+, 1/2^+)$
				Chiral T
				$H'''(\tilde{H}_{\alpha\dot{2}}) (1/2, 1/2^-, 0)$
				$\Phi'''(B^{i\dot{2}}, C) (0, , 1^0, 0) \oplus (0, 0, 1^0)$
				$\psi_{\alpha\dot{2}}^2 \equiv \lambda_{\alpha}^2 (1/2, 1/2^-, 1/2^-)$
				$\left. \begin{matrix} \tilde{\psi}_{\dot{2}2}^1 \\ \tilde{\psi}_{i2}^1 \end{matrix} \right\} \equiv \bar{\lambda}_{\dot{\alpha}}^2 (0, 1^-, 1/2^+)$
				$(0, 1^0, 1/2^+) \oplus (0, 0, 1/2^+)$

(4.8)

in which $\Phi'(\phi, \tilde{\phi})$ etc. show that Φ' is a complex field made of two real fields $\phi, \tilde{\phi}$, and $(\dots) \oplus (\dots)$ indicates one of the linear combinations of two states or both.

One may give a mass to the hypermultiplet so that the $N = 4$ supersymmetry reduces to $N = 1$ through $N = 2$. However this way would break the $SU(2)$ symmetry of the three chiral multiplets in $N = 1$ declared by Vafa and Witten [7], and the Gh-duality would be lost as well. So we shall consider the case in which the reduction is direct. The symmetry between three chiral multiplets thus can be preserved and the Gh-duality can also be inherited as follows. Replace Φ' and Φ'' by their complex combinations ϕ and $\tilde{\phi}$ so that they are Gh-dual to each other (we use the same symbols ϕ and $\tilde{\phi}$ to show that they have the same behavior under the Gh-duality transform). Then there is also a symmetry between the superfields U and V under the Gh-duality (to avoid unnecessary complexity we have neglected the variation of auxiliary fields).

One may think to construct Gh-selfdual and anti-selfdual multiplets out of the gauge multiplet and the T multiplet. However they will not bring us with new information. So we prefer to use the gauge multiplet and the T multiplet and their respective dual multiplets while computing the mass perturbation.

When $B^{i\dot{2}} = C = 0$, the multiplets in (4.8) will degenerate to two Gh-dual $N = 2$ massless multiplets or two groups of $N = 1$ Gh-dual gauge multiplets and chiral multiplets. Especially the Gh-dual complex fields ϕ and $\tilde{\phi}$ are complex conjugate to each other as well.

5 The mass term

Follow the analysis of supersymmetry mentioned above, a mass-like perturbation term preserving $N = 1$ supersymmetry we can add is

$$\Delta L = -\frac{1}{2} \int_M \sqrt{g} d^4 x d^2 \theta \text{Tr}(m_1 U^2 + m_2 V^2 + m_3 T^2) - h.c. \quad (5.1)$$

in which two types of contribution should be considered:

- The first two terms with $m_1 = m_2 = m$ while the “mass” can be replaced by a holomorphic $(2, 0)$ form on a Kähler manifold on which $H^{2,0}(M) \neq 0$. This is perhaps the case of “trivial embedding” in ref.[7].
- The last term with m_3 a holomorphic two form. This looks like the “irreducible embedding” case [7].

Because we could not construct an observable from the superfield T (which could contribute to the partition function) with balanced ghost number, we have to consider only the first case.

By imitating an analogous discussion made by Witten, we can easily prove that when the desired vanishing theorem holds the mass term of chiral superfields U and V are equivalent to

$$\sum_{A=1,2} \sum_a \alpha_a I_A(\omega) \quad (5.2)$$

up to \hat{Q}_A -exact terms, where $I_A(\omega) = \int_M \mathcal{O}_A^{(2)} \wedge \omega$ are our friends, observables for $k = 2$, and ω is a nonvanishing holomorphic two form related to m . In fact, Witten chose

$$m = \sigma_{mn\dot{2}\dot{2}} \omega_{kl} \epsilon^{mnkl} \quad (5.3)$$

so that (let $\Phi_A = U, V$)

$$\begin{aligned} \Delta L &= -\frac{1}{2} \sum_A \int_M \omega_{kl} dx^k \wedge dx^l d^2 z d^2 \theta \text{Tr} \Phi_A^2 \\ &= -\frac{1}{4} \sum_A \int_M \epsilon^{\alpha\beta} \sigma_{m\alpha\dot{2}} \sigma_{n\beta\dot{2}} \omega_{kl} dx^m \wedge dx^n \wedge dx^k \wedge dx^l \text{Tr} \Phi_A^2|_{\theta\theta} \\ &= -\frac{1}{8} \sum_A \int_M \sqrt{g} d^4 x \text{Tr}(m \psi_{A\alpha} \psi^{A\alpha} + \bar{m} \bar{\psi}_{A\dot{\alpha}} \bar{\psi}^{A\dot{\alpha}} + e^2 \sqrt{2} m \bar{m} \text{Tr}(\tilde{\phi}\phi), \end{aligned} \quad (5.4)$$

which equals to

$$\begin{aligned} &-\frac{1}{2} \int \sqrt{g} d^4 x \text{Tr} \psi_{A\alpha} \psi^{A\alpha} \bar{\sigma}_{mn\dot{2}\dot{2}} \omega_{kl} \epsilon^{mnkl} + \{\hat{Q}_A, V^A\} \\ &\sum_A I_A(\omega) + \{\hat{Q}_A, V^A + \dots\} \end{aligned} \quad (5.5)$$

$$(5.6)$$

with

$$\begin{aligned} V^1 &= -\frac{1}{4} \int \sqrt{g} d^4 x \text{Tr} \tilde{\phi} \bar{\psi}^{\dot{\alpha}} \epsilon^{\beta\dot{2}} \bar{\sigma}_{np\dot{\alpha}\dot{\beta}} \bar{\omega}_{kl} \epsilon^{npkl}, \\ V^2 &= -\frac{1}{4} \int \sqrt{g} d^4 x \text{Tr} \phi \bar{\chi}^{\dot{\alpha}} \epsilon^{\beta\dot{2}} \bar{\sigma}_{np\dot{\alpha}\dot{\beta}} \bar{\omega}_{kl} \epsilon^{npkl}, \end{aligned} \quad (5.7)$$

and the last transformation law in (4.7) should be changed into

$$\begin{aligned}\delta\bar{\psi}^{\dot{\alpha}} &= -\frac{1}{2}e^2\rho^1\phi\epsilon^{\dot{\alpha}\dot{2}}\bar{\sigma}_{np\dot{2}\dot{2}}\omega_{kl}\epsilon^{npkl}, \\ \delta\bar{\chi}^{\dot{\alpha}} &= -\frac{1}{2}e^2\rho^2\tilde{\phi}\epsilon^{\dot{\alpha}\dot{2}}\bar{\sigma}_{np\dot{2}\dot{2}}\omega_{kl}\epsilon^{npkl}.\end{aligned}\quad (5.8)$$

Hence we have

$$L + \Delta L = L + \sum_A (I_A(\omega) + \{\hat{Q}_A, \dots\}). \quad (5.9)$$

Considering the requirement of ghost number balance, the perturbed correlation function becomes

$$\begin{aligned}&\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_1(x_r) I_1(\Sigma_1) \dots I_1(\Sigma_1) e^{I_1(\omega)} \rangle_1 \\ &\times \langle \mathcal{O}_2(x_{r+1}) \dots \mathcal{O}_2(x_{r+r'}) I_2(\Sigma_{s+1}) \dots I_2(\Sigma_{s+s'}) e^{I_2(\omega)} \rangle_2.\end{aligned}\quad (5.10)$$

Again, with similar discussions as in [5], the expression for the correlation functions goes back to the key formula (3.14). If the vanishing theorem is valid, the $N = 4$ supersymmetry can be viewed as a pair of $N = 2$ supersymmetric theories which further decompose into $N = 1$ ones through perturbation. Consequently we now have gluino condensation both in regular multiplet and its Gh-dual multiplet,

$$\langle \lambda_{1\alpha} \lambda_1^\alpha \rangle = (\mu_1)^3, \quad \langle \lambda_{2\alpha} \lambda_2^\alpha \rangle = (\mu_2)^3, \quad (5.11)$$

where μ_1, μ_2 are mass scale renormalization parameters.

Since the relevant theories reduce to the minimal $N = 1$ systems, the perturbation leads to a dynamically generated mass gap.

Because of the mass term, the theory now has the $Z_4 \times Z'_2$ symmetry as shown in ref.[5], where the Z'_2 with generator β is the symmetry which transforms Φ_A to $-\Phi_A$, which will be used in the next section.

6 Polynomial invariants

To get the formulae on general Kähler manifolds, Witten's routine requires that one first neglect the difference between physical and topological theories, i.e. consider the hyper-Kähler case, then make a correction involving the twisting and the canonical divisor of M , when the canonical class of M is nontrivial.

We first derive the expressions on hyper-Kähler manifolds. The global symmetry group $Z_4 \times Z'_2$ now is broken to $Z_2 \times Z'_2$ with double degeneracy of the vacuum. The two vacuum states are denoted by $|+\rangle$ and $|-\rangle$. R -symmetry (Z_4 with generator α) tells us that for both $A = 1, 2$,

$$\alpha : \psi_{A\alpha} \rightarrow i\psi_{A\alpha}, \quad \bar{\psi}_{\dot{\alpha}}^A \rightarrow -i\bar{\psi}_{\dot{\alpha}}^A, \quad \mathcal{O}_A \rightarrow -\mathcal{O}_A, \quad \eta_A \rightarrow -\eta_A. \quad (6.1)$$

The last transition is due to that only the $I_A^{(1,1)}$ part of I_A contributes [5]. The zero modes of ψ_A minus $\bar{\psi}^A$ is

$$\Delta = 4(k - \nu) + \nu = \frac{1}{2}\dim\mathcal{M}, \quad (6.2)$$

in which

$$\nu \equiv \frac{\chi + \sigma}{4} = \text{integer on the Kähler manifolds.} \quad (6.3)$$

Due to the opposite ghost number in the two parts of the correlation function, we know

$$C_-^1 = i^\Delta C_+^1 = i^\nu C_+^1, \quad C_-^2 = i^{-\Delta} C_+^2 = i^{-\nu} C_+^2. \quad (6.4)$$

In addition, the canonical divisor on hyper-Kähler manifolds vanishes,

$$K \bullet K = 2\chi + 3\sigma = 0. \quad (6.5)$$

So we can choose

$$C_+^A = e^{a_A \nu}. \quad (6.6)$$

As a result, the generating functional for the correlation function can be written as (set $\eta_A \equiv \eta_{A+}, o_A = \langle \mathcal{O}_A \rangle_+$)

$$\begin{aligned} & \langle \exp \sum_A \left(\sum_a \alpha_{Aa} I_A(\Sigma_a) + \lambda_A \mathcal{O}_A \right) \rangle \\ &= e^{(a_1+a_2)\nu} \left\{ \exp \left(\frac{1}{2} \eta_1 \sum_{a,b} \alpha_{1a} \alpha_{1b} \#(\Sigma_a \cap \Sigma_b) + \lambda_1 o_1 \right) \right. \\ & \quad \left. + i^\nu \exp \left(-\frac{1}{2} \eta_1 \sum_{a,b} \alpha_{1a} \alpha_{1b} \#(\Sigma_a \cap \Sigma_b) - \lambda_1 o_1 \right) \right\} \\ & \quad \times \left\{ \exp \left(\frac{1}{2} \eta_2 \sum_{a,b} \alpha_{2a} \alpha_{2b} \#(\Sigma_a \cap \Sigma_b) + \lambda_2 o_2 \right) \right. \\ & \quad \left. + i^{-\nu} \exp \left(-\frac{1}{2} \eta_2 \sum_{a,b} \alpha_{2a} \alpha_{2b} \#(\Sigma_a \cap \Sigma_b) - \lambda_2 o_2 \right) \right\} \Big|_0. \end{aligned} \quad (6.7)$$

Next we consider the general Kähler case. By general case we mean the Kähler manifold on which $H^{2,0} \neq 0$ and nonzero canonical divisor C exists. Assume also $C = \bigcup_y C_y$ is the union of smooth, disjoint Riemann surfaces C_y along which ω has simple zeros. This brings us with the contribution of the so-called cosmic string [5].

Because we have the symmetry breaking from $Z_2 \times Z'_2$ down to Z_2'' , a diagonal subgroup generated by the operator $(-1)^F$ in which F counts the fermion number, the vacua $|\pm\rangle$ bifurcate into four states $|\pm, +\rangle$ and $|\pm, -\rangle$ near the cosmic string [5]. Meanwhile, there are contributions of $\sum_y \#(\Sigma \cap C_y) V_{Ay}$ in which $\#(\Sigma \cap C_y)$ is the algebraic intersection number between Σ and C_y , V_{Ay} are local operators V_A inserted on C_y which comes from the integration $I_A(\Sigma)$. Since the operators V_A transform like $I_A^{(1,1)}$ under the group $Z_4 \times Z'_2$, i.e. $\alpha V_A = i V_A, \beta V_A = -V_A$, their expectation values in four vacua are related in the form

$$\langle V_A \rangle_{++} = i \langle V_A \rangle_{-+} = -\langle V_A \rangle_{+-} = -i \langle V_A \rangle_{--} = v_A.$$

Moreover, the partition functions acquire one more factor [7],

$$e^{\frac{b_A}{2} \chi(C_y)} = e^{b_A(1-g_y)} = e^{-b_A(2\chi+3\sigma)(M)}, \quad (6.8)$$

in which g_y is the genus of C_y .

In conclusion, to the approximation we have used, the final generating functional of $N = 4$ topological model on Kähler manifold reads

$$\begin{aligned}
& \langle \exp \sum_A \left(\sum_a \alpha_{Aa} I_A(\sigma_a) + \lambda_A \mathcal{O}_A \right) \rangle \\
&= e^{(a_1+a_2)\nu} \left\{ \exp \left(\frac{1}{2} \eta_1 \sum_{a,b} \alpha_{1a} \alpha_{1b} \#(\Sigma_a \cap \Sigma_b) + \lambda_1 \mathcal{O}_1 \right) \right. \\
&\quad \times \prod_y e^{b_1(1-g_y)} (e^{\Phi_{1y}} + t_y e^{-\Phi_{1y}}) \\
&\quad + i^\nu \exp \left(-\frac{1}{2} \eta_1 \sum_{a,b} \alpha_{1a} \alpha_{1b} \#(\Sigma_a \cap \Sigma_b) - \lambda_1 \mathcal{O}_1 \right) \\
&\quad \times \prod_y e^{b_1(1-g_y)} (e^{-i\Phi_{1y}} + t_y e^{i\Phi_{1y}}) \Big\} \\
&\times \left\{ \exp \left(\frac{1}{2} \eta_2 \sum_{a,b} \alpha_{2a} \alpha_{2b} \#(\Sigma_a \cap \Sigma_b) + \lambda_2 \mathcal{O}_2 \right) \right. \\
&\quad \times \prod_y e^{b_2(1-g_y)} (e^{\Phi_{2y}} + t_y e^{-\Phi_{2y}}) \\
&\quad + i^{-\nu} \exp \left(-\frac{1}{2} \eta_2 \sum_{a,b} \alpha_{2a} \alpha_{2b} \#(\Sigma_a \cap \Sigma_b) - \lambda_2 \mathcal{O}_2 \right) \\
&\quad \times \prod_y e^{b_2(1-g_y)} (e^{-i\Phi_{2y}} + t_y e^{i\Phi_{2y}}) \Big\} \Big|_0, \tag{6.9}
\end{aligned}$$

in which

$$\Phi_{Ay} = \sum_a \alpha_{Aa} \#(\Sigma_a \cup C_y) v_A \tag{6.10}$$

and

$$t_y = (-)^{\epsilon_y}, \tag{6.11}$$

$\epsilon_y = 0, 1$ depending on whether the spin structure is even or odd, and [5]

$$\sum_y \epsilon_y = \Delta \mod 2 = \nu \mod 2. \tag{6.12}$$

7 Concluding remarks

We have developed the correlation functions for $N = 4$ topological model using the approximation of factorization. The topological invariants thus obtained are closely related to the Donaldson invariants (the product of Donaldson invariants). However, if we go further to calculate the corrections (for example, assume the vanishing theorem fails to hold on some manifolds), we might find some topological invariants entirely different from Donaldson. Can such invariants bring us new information in differential geometry? What are their mathematical implications? These questions may be interesting for further studies.

Another interesting question is to test the modular properties of the correlation functions and get more evidences for the S -duality [7, 14] when the θ -angle is taken into consideration. One

may guess that all of the universal constants in our formulas should become modular functions of $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ just like in the partition function analyzed by Vafa and Witten [7]. But it might be very difficult to find the constraints imposed by modular transformations.

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